

Strong Coupling Expansion of the Entanglement Entropy of Yang-Mills Gauge Theories

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Abstract

We calculate the entanglement entropy of the $SU(N)$ Yang-Mills gauge theories on the lattice under the strong coupling expansion in powers of $\beta = 2N/g^2$, where g is the coupling constant. Using the replica method, our Lagrangian formalism maintains gauge invariance on the lattice. At $O(\beta^2)$ and $O(\beta^3)$, the entanglement entropy is solely contributed by the central plaquettes enclosing the conical singularity of the n -sheeted Riemann surface. The area law emerges naturally to the highest order $O(\beta^3)$ of our calculation. The leading $O(\beta)$ term is negative, which could in principle be canceled by taking into account the “cosmological constant” living in interface of the two entangled subregions. This unknown cosmological constant resembles the ambiguity of edge modes in the Hamiltonian formalism. We further speculate this unknown cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Furthermore, it could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

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I. INTRODUCTION

Entanglement entropy is a measure of the level of entanglement between the degrees of freedom in two subregions of a physical system. Besides being a fundamental and mysterious property of quantum mechanics and quantum field theory, quantum entanglement is of practical use. In some systems, entanglement entropy plays the role of an order parameter to characterize quantum phase transitions [1–3], while in others it demonstrates the scaling behavior [4, 5] near the critical point. In field theory, a widely used method for calculating the entanglement entropy is the replica trick [6]. This method calculates the trace of the reduced density matrix to the n -th power in the path integral formalism, which amounts to computing the free energy of the system on a n -sheeted Riemann surface, or equivalently on a “cone” with a conical angle of $2n\pi$. The entanglement entropy is then obtained as a response of the free energy to the change of the conical angle at $n = 1$. This is similar to calculating the black hole entropy by evaluating the response of the free energy of fields with respect to the deficit angle in the Euclidean spacetime [7].

In previous studies, while the computation of entanglement entropy for the scalar and the spinor fields are considered straight forward, it becomes more subtle for the gauge fields. It was found by Kabat [7] that the gauge fields in the black hole entropy method yields an extra negative contact term compared to their entanglement entropy. This term is due to the interaction of the particles with the horizon, and is believed to be related to the choice of the boundary condition while removing the tip of the cone due to the black hole in the Euclidean space. It was later interpreted by [8] as arising from incorrect treatment of the zero modes.

As for the gauge fields on the lattice, the entanglement entropy had been studied with the Hamiltonian [9–14] and the Lagrangian [15–17] approaches. In the Hamiltonian approach for the case of the gauge fields on the lattice [9–12], one needs to impose the Gauss law or the gauge fixing constraints in order to get rid of the unphysical degrees of freedom, and the ambiguity arises in determining which subregion the gauge links on the boundary belong to. This leads to the difficulty in decomposing the global gauge invariant states into the direct products of those living in each subregions. It was proposed [9] that these ambiguities might be compensated by edge modes living in the interface of the subregions and determined by the transverse electric fields. This echoes the concept that the contact

term in [7] arises from the sources on the horizon. In [9] this negative contact term was found to arise from the entanglement of the edge modes. Moreover, in order to overcome the puzzle of Hilbert space decomposition, [11] employs various choices of electric, magnetic, and trivial centers in the operator algebra in the Hamiltonian formalism, among which the trivial center is related to some spacial gauge fixing choice, and hence the entanglement entropy thus obtained is not gauge invariant. Besides, the electric and magnetic choices also result in different tripartite topological entanglement entropy. In view of these ambiguities, we explore a complementary approach—the Lagrangian formalism—to shed light on the problem from a different perspective.

The replica method arises from taking the derivative of the trace of n copies of the reduced density matrix by n . Since the density matrix can always be expressed in the path integral formalism as long as the quantum field theory is local, the replica method is valid for the gauge fields at continuum. We start from the replica method at continuum, and then discretize the spacetime into a squared lattice. We divide an infinitely large system into two semi-infinite subregions by a flat boundary, and decompose the spacetime into a direct product of a 2-dimensional cone with a conical angle of $2n\pi$ (or equivalently an n -sheeted Riemann surface) and an ordinary Euclidean space transverse to cone. Then we use the Wilson gauge action which has the advantage of gauge invariance on a discrete lattice. This action sums over the Wilson loops on the plaquettes, including those on the n -sheeted Riemann surface and those on the ordinary Euclidean space. In contrast with the previous studies where the conical singularity is placed on the lattice site, we use a different discretization setup by locating the conical singularity in the center of the plaquette. As a result, the branch lines on the n -sheeted Riemann surface cut across the links, and there is no lattice site on the cut. (See Fig. 1.) This setup yields two types of plaquettes on the squared lattice: the central plaquettes encircling the tip of the cone, formed by $4n$ links, and the regular plaquettes formed by 4 links with no singularity inside. When the conical angle, or n , changes, only the central plaquettes (i.e. those plaquettes with $4n$ links) respond to this change. As a result, entanglement entropy necessarily involves those central plaquettes. The fact that all of the central plaquettes live across the interface between the two subregions naturally give rise to the area law, which states that the leading contribution to entanglement entropy scales as the area of the interface.

The connection to the area law can be further demonstrated order by order diagram-

matically under the strong coupling expansion. Interestingly, we find the leading term in the strong coupling expansion negative. However, symmetries of the action allow a two-dimensional cosmological constant living in the interface [18] which could provide a positive contribution at an even lower order. We speculate that the freedom to tune this unknown two-dimensional cosmological constant corresponds to the ambiguities encountered in the Hamiltonian approach. We further speculate that the two-dimensional cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. They can play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

This paper is organized as follows. In Sec. II, we briefly review the notion of entanglement entropy and the replica method. The entanglement entropy of Yang-Mills fields on the lattice under the strong coupling expansion is calculated in Sec. III, and the cancellation of the negative contribution by including a 2 dimensional cosmological constant is discussed in Sec. IV. Sec. V compares our result with the previous ones obtained by the Hamiltonian methods, and Sec. VI summarizes our study.

II. ENTANGLEMENT ENTROPY AND THE REPLICA METHOD

Suppose our system occupies an infinitely large and flat $d + 1$ dimensional spacetime and is divided into two semi-infinite subregions A and B . They are divided by an infinite and flat $d - 1$ dimensional space-like boundary. The entanglement entropy (EE) of a quantum theory between the two subregions is defined by the von Neumann entropy. With some simple algebra, it can be re-expressed as:

$$S_{\text{EE}} = -\text{tr}[\rho_A \ln \rho_A] = - \left. \frac{\partial}{\partial n} \right|_{n \rightarrow 1} \ln \text{tr}[\rho_A^n] \quad (1)$$

where $\rho_A = \text{tr}_B[\rho]$ is the reduced density matrix by tracing out the degrees of freedom in region B . This expression is called the replica method because it involves n copies of ρ_A .

An elegant path integral formulation to compute the entanglement entropy using the replica method was first introduced in [6] (see also [19]). In this set up, one recalls that $\rho_{ij} = \langle i | e^{-H/T} | j \rangle$ and $\text{Tr}[\rho]$ is the partition function calculated in finite temperature field theory with appropriate boundary conditions (periodic and anti-periodic boundary conditions for bosons and fermions, respective) imposed for fields at Euclidean time $\tau = 0$ and $1/T$, where

T is the temperature. Then $\text{Tr}[\rho^2]$ can be computed by doubling the period (by imposing appropriate boundary conditions at $\tau = 0$ to $2/T$). Similarly, $\text{Tr}[\rho_A^2]$ is computed by doubling the period (from 0 to $2/T$) for region A while region B still has the single period (from 0 to $1/T$) as shown in the left plot of Fig. 1(A), which is equivalent to performing the path integral on a 2-sheeted Riemann surface in the right plot. One can generalize this set up to $\text{Tr}[\rho_A^n]$ for an arbitrary n . There is no restriction on the space partition between A and B . The sizes of A , B , and T can be either finite or infinite.

In this paper, we will just concentrate on the simplest case with the sizes of space and (Euclidean) time to be both infinite (i.e. $T = 0$) and the interface between A and B to be a flat infinite plane. In this limit, the n -sheeted Riemann surface has a conical structure as shown in Fig. 1(B) with the time and longitudinal spacial direction (the direction that is perpendicular to the interface) lying on the cone while the space on the interface transverse to the cone.

As a result, $\text{tr}[\rho_A^n]$ becomes a partition function Z_n on the n -sheeted Riemann surface, or, in our case, a cone with $2n\pi$ conical angle, normalized by n -copies of the partition function on the ordinary Euclidean space Z_1^n :

$$\text{tr}[\rho_A^n] = \frac{Z_n}{Z_1^n}, \quad (2)$$

which ensures that as $n = 1$, $\text{tr}[\rho_A^n] = 1$. The entanglement entropy is then given by

$$S_{\text{EE}} = - \left. \frac{\partial}{\partial n} (\ln Z_n - n \ln Z_1) \right|_{n \rightarrow 1} \stackrel{n=1+\epsilon}{=} -\frac{1}{\epsilon} [\ln Z_{1+\epsilon} - (1 + \epsilon) \ln Z_1]_{\epsilon \rightarrow 0}. \quad (3)$$

Note that n is taken as an integer in the integral of Z_n . After one obtains the analytic expression for $\text{tr}[\rho^n]$, then n can be analytically extended to non-integers to carry out the differentiation at $n = 1$.

III. CALCULATION ON AN N-SHEETED LATTICE MANIFOLD

We now discrete a $d + 1$ dimensional spacetime into a squared lattice. The spacetime is decomposed into a direct product of a $1 + 1$ dimensional n -sheeted lattice (which we call the parallel dimensions) with coordinates (x_{\parallel}, τ) and a discrete $d - 1$ dimensional transverse space labelled by x_{\perp} . We discretize the n -sheeted Riemann surface in such a way that the end point of the cut on each sheet (or, the conical singularity corresponding to the tip of

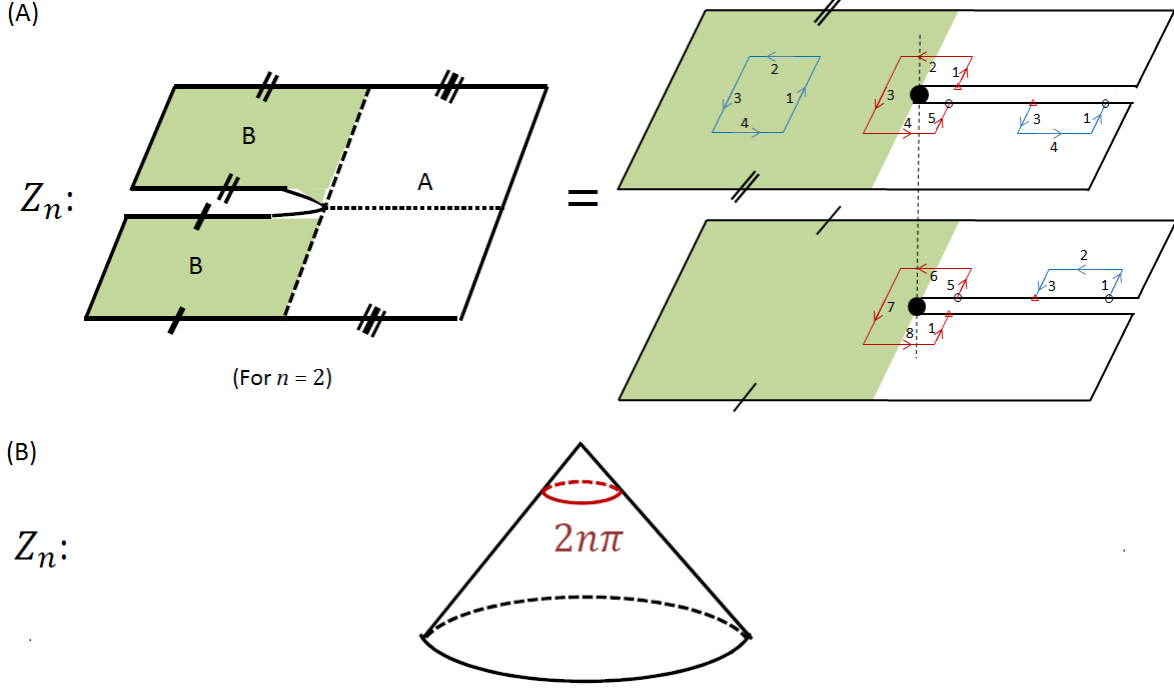


Figure 1: The n -sheeted Riemann surface in the replica trick. (A) illustrates the equivalent geometric structures of the n -sheeted manifold in the case of $n = 2$ arising from the replica trick, on which the partition function Z_n is computed. The unshaded and shaded parts label the subregion A and B, respectively. The subregion B is traced out in the reduced density matrix ρ_A . The 2 sheets on the right are at the same x_\perp coordinates, attaching to each other at the conical singularity which is represented by the black dot and the dotted vertical line. In the right figure, it is also demonstrated explicitly after discretization a central plaquette with $4n$ edges encircling the conical singularity, and the ordinary plaquettes with 4 links located on a sheet and across two adjacent sheets (see also Fig. 2). The numbers label the order of the link variables to form the plaquettes. When the sizes of space and (Euclidean) time are both infinite (i.e. $T = 0$), this geometry is equivalent to a cone with $2n\pi$ conical angle depicted in (B).

the cone) locates inside a plaquette rather than sitting on a lattice site. It will be clear later that our result is independent of the location of the conical singularity as long as it is encircled by a plaquette. For simplicity we choose that the conical singularity sits in the center of a plaquette.

As a result of the discretization, there are two types of plaquettes. The *ordinary plaquettes* contain no conical singularity inside, and are composed of 4 links. They make up

all plaquettes in the transverse dimensions and most plaquettes in the parallel dimensions, and their plaquette variables are denoted by $U_{\square}^{(k)} = \mathcal{P} \prod_{l \in \square} U_l^{(k)}$, where k indicates their location bring on the k -th sheet, and \mathcal{P} signifies the ordered product of the link variables $U_l^{(k)}$ forming the plaquette. On the other hand, the *central plaquettes* enclose the conical singularity on each sheet in the parallel dimensions and are composed of $4n$ links. Their plaquette variables are denoted by $U_{\square}(x_{\perp}) = \mathcal{P} \prod_{l \in \square} U_l$, where x_{\perp} is the location of the parallel planes in the transverse dimensions. See Fig. 2 for the cartoon of these two kinds of plaquettes.

Recall that the partition function of the lattice gauge theory on a one-sheet manifold is given by

$$Z = \int \mathcal{D}U \exp \left\{ -\beta \sum_{\square} \left[1 - \frac{1}{N} \text{Re tr} U_{\square} \right] \right\} \xrightarrow{a \rightarrow 0} \int \mathcal{D}A \exp \left\{ - \int d^4x \left[\frac{1}{4g^2} \text{tr} F^2 \right] \right\}. \quad (4)$$

where $\beta = 2N/g^2$ and \square labels the location of plaquettes. The plaquette variable U_{\square} is the local Wilson loop composed of the ordered product of four gauge links, $U_{\square} = \mathcal{P} \prod_{l \in \square} U_l$ where \mathcal{P} indicates the ordered product and U_l is the link variable representing the gauge fields. The action recovers the Yang-Mills action in the continuum limit by setting the lattice spacing $a \rightarrow 0$.

To construct a lattice gauge field system in a general $d + 1$ dimensions whose $1 + 1$ dimensions is an n -sheeted manifold, we rewrite the partition function in Eq.(4) in terms of the ordinary plaquettes $U_{\square}^{(k)}$ and the central ones $U_{\square}(x_{\perp})$, such that the partition function reads

$$\begin{aligned} Z_n &= \int \left[\prod_{m=1}^n \mathcal{D}U^{(m)} \right] \exp \left\{ \frac{\beta}{N} \sum_{k=1}^n \sum_{\square} \text{Re tr} [U_{\square}^{(k)} - 1] + \frac{\beta}{nN} \sum_{\mathbf{x}_{\perp}} \text{Re tr} [U_{\square}(\mathbf{x}_{\perp}) - 1] \right\} \quad (5) \\ &\xrightarrow{a \rightarrow 0} \int \left[\prod_{m=1}^n \mathcal{D}A^{(m)} \right] \exp \left\{ -\frac{1}{4g^2} \int d^2x_{\perp} \left(\sum_{k=1}^n \int_{\mathbb{R}^2 - \{0\}} d^2x_{\parallel} \text{tr} F^{(k)2} \right. \right. \\ &\quad \left. \left. + \frac{1}{n} \sum_{k,l=1}^n \int_{\{0\}} d^2x_{\parallel} \text{tr} [F^{(k)} F^{(l)}] \right) \right\} \\ &= \int \left[\prod_{m=1}^n \mathcal{D}A^{(m)} \right] \exp \left\{ -\frac{1}{4g^2} \sum_{k=1}^n \int d^2x_{\perp} \int d^2x_{\parallel} \text{tr} F^{(k)2} \right\}, \end{aligned}$$

where we have assumed the boundary condition

$$F^{(k)}(\mathbf{x}_{\perp}, \mathbf{x}_{\parallel} = \mathbf{0}) = F^{(l)}(\mathbf{x}_{\perp}, \mathbf{x}_{\parallel} = \mathbf{0}), \quad (k \neq l), \quad (6)$$

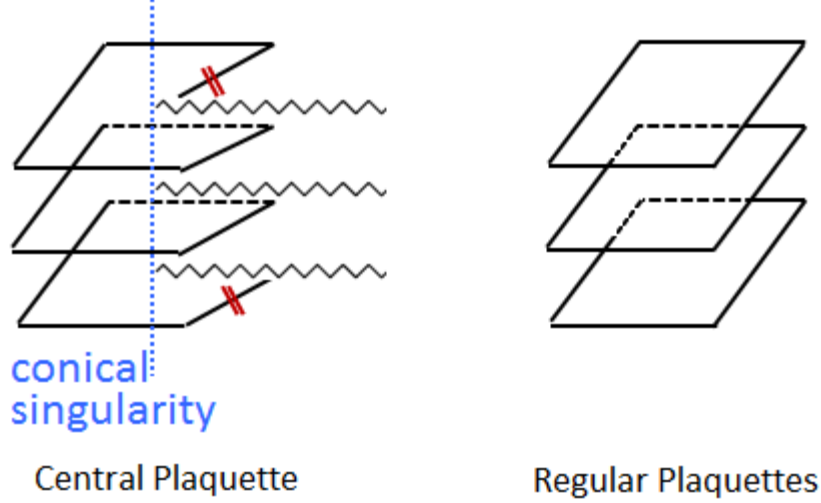


Figure 2: One central plaquette (left) enclosing the conical singularity v.s. three regular plaquettes on three different sheets (right) in the case of $n = 3$ sheets. In the left figure, the dotted line represents the conical singularity to which all three sheets attach, and the wavy lines are cuts on each sheet. The central plaquette has 12 edges and it takes 6π to go around it, while each regular plaquette has 4 links.

i.e. the field strength on each sheet is assumed be equal. This condition is natural if we impose 2π rotation symmetry on the n -sheeted surface. We have introduced an extra $1/n$ factor to the central plaquette terms, because the $4n$ gauge links of the central plaquettes encircle the F_{01} flux over an area na^2 , and give rise to a factor $\propto n^2 a^4 F_{01}^2$ to the action, inconsistent with the contribution from the transverse plaquettes scaling as na^4 . Therefore a factor of $1/n$ is introduced to compensate this effect.

Expanding Eq.(5) to the second order of β , we have

$$\begin{aligned}
Z_n = e^{-\beta n \mathcal{N} - \frac{\beta}{n} \mathcal{N}_\perp} \int \left[\prod_{m=1}^n \mathcal{D}U^{(m)} \right] \left\{ 1 + \frac{\beta}{N} \sum_{k=1}^n \sum_{\square} \text{Re tr} [U_{\square}^{(k)}] + \frac{\beta}{nN} \sum_{\mathbf{x}_\perp} \text{Re tr} [U_{\square}(\mathbf{x}_\perp)] \right. \\
+ \frac{\beta^2}{2N^2} \sum_{k,l=1}^n \sum_{\square, \square'} [\text{Re tr} U_{\square}^{(k)}] [\text{Re tr} U_{\square'}^{(l)}] \\
\left. + \frac{\beta^2}{2N^2} \frac{1}{n^2} \sum_{\mathbf{x}_{1\perp}, \mathbf{x}_{2\perp}} [\text{Re tr} U_{\square}(\mathbf{x}_{1\perp})] [\text{Re tr} U_{\square}(\mathbf{x}_{2\perp})] + \text{cross-term} + O(\beta^3) \right\}. \quad (7)
\end{aligned}$$

Here \mathcal{N} is the number of plaquettes on a single sheet for plaquettes not encircling the conical singularity, including both the parallel and transverse plaquettes. \mathcal{N}_\perp is the number of plaquettes encircling the conical singularity.

With $\text{Retr}U = \frac{1}{2}(\text{tr}U + \text{tr}U^\dagger)$, we first evaluate the leading contribution of the Wilson loops from $\int \mathcal{D}U \text{tr}U \text{tr}U^\dagger$:

$$\begin{aligned} Z_n &= e^{-\beta n \mathcal{N} - \frac{\beta}{n} \mathcal{N}_\perp} \left\{ 1 + \frac{\beta^2}{2N^2} \frac{1}{2} \sum_{k,l=1}^n \sum_{\square, \square'} \int \left[\prod_{m=1}^n \mathcal{D}U^{(m)} \right] \left[\text{tr}U_{\square}^{(k)} \right] \left[\text{tr}U_{\square'}^{(l)\dagger} \right] \right. \\ &\quad \left. + \frac{\beta^2}{2N^2} \frac{1}{2} \frac{1}{n^2} \sum_{\mathbf{x}_{1\perp}, \mathbf{x}_{2\perp}} \int \left[\prod_{k=1}^n \mathcal{D}U^{(k)} \right] \left[\text{tr}U_{\square}(\mathbf{x}_{1\perp}) \right] \left[\text{tr}U_{\square}^\dagger(\mathbf{x}_{2\perp}) \right] + O(\beta^3) \right\} \\ &= e^{-\beta n \mathcal{N} - \frac{\beta}{n} \mathcal{N}_\perp} \left\{ 1 + \frac{\beta^2}{4N^2} n \mathcal{N} + \frac{\beta^2}{4N^2} \frac{1}{n^2} \mathcal{N}_\perp + O(\beta^3) \right\}. \end{aligned} \quad (8)$$

Here we have used the Haar measure

$$\int dU U_{ij} = 0 = \int dU U_{ij}^\dagger, \quad (9)$$

$$\int dU U_{ij} U_{kl}^\dagger = \frac{1}{N} \delta_{il} \delta_{jk}, \quad (10)$$

such that

$$\int \mathcal{D}U \text{tr}U_{\square}^{(k)} \text{tr}U_{\square}^{(l)\dagger} = \delta_{kl}, \quad (11)$$

$$\int \mathcal{D}U \text{tr}U_{\square}(\mathbf{x}_{1\perp}) \text{tr}U_{\square}^\dagger(\mathbf{x}_{2\perp}) = \delta_{x_1 x_2}. \quad (12)$$

Note that (11) and (12) are independent of n . Taking the combination

$$\ln Z_n - n \ln Z_1 = -\beta \mathcal{N}_\perp \left[\frac{1}{n} - n \right] + \frac{\beta^2 \mathcal{N}_\perp}{4N^2} \left(\frac{1}{n^2} - n \right) + O(\beta^3), \quad (13)$$

and then one obtains the Renyi entropy

$$S_n = -\beta \mathcal{N}_\perp \frac{1+n}{n} + \frac{\beta^2 \mathcal{N}_\perp}{4N^2} \left(\frac{1+n+n^2}{n^2} \right) + O(\beta^3). \quad (14)$$

This means no matter how we place the lattice, as long as the conical singularity inside the central plaquettes (i.e. the central plaquettes exist), the result will be the same. But if one chooses to put the conical singularity on a lattice site with no plaquette encircling the conical singularity, then there is no central plaquettes to offer the $\frac{1}{n}$ and $\frac{1}{n^2}$ factor in (13), but the factors n appears instead and effect of $\ln Z_n$ is canceled by that from $n \ln Z_1$.

Another non-trivial contribution from the Wilson loops is $\int \mathcal{D}U (\text{tr}U)^N$. Using the identities

$$\int dU U_{i_1 j_1}^{(p)} U_{i_2 j_2}^{(q)} \dots U_{i_N j_N}^{(s)} = \frac{1}{N!} \epsilon_{i_1 i_2 \dots i_N} \epsilon_{j_1 j_2 \dots j_N} \delta^{pq \dots s}, \quad (15)$$

$$\int dU U_{i_1 j_1}^{(p)\dagger} U_{i_2 j_2}^{(q)\dagger} \dots U_{i_N j_N}^{(s)\dagger} = \frac{1}{N!} \epsilon_{i_1 i_2 \dots i_N} \epsilon_{j_1 j_2 \dots j_N} \delta^{pq \dots s}, \quad (16)$$

one finds such contribution are of $O(\beta^N)$:

$$\int \mathcal{D}U (\text{tr} U_{\square})^N = 1, \quad (17)$$

$$\int \mathcal{D}U [\text{tr} U_{\square}(\mathbf{x}_{\perp})]^N = 1, \quad (18)$$

$$\begin{aligned} Z_n^{\beta^N} &= \int \left[\prod_{m=1}^n \mathcal{D}U^{(m)} \right] \frac{\beta^N}{N!N^N} \left\{ \sum_{k=1}^n \sum_{\square} \text{Re tr} [U_{\square}^{(k)}] + \frac{1}{n} \sum_{\mathbf{x}_{\perp}} \text{Re tr} [U_{\square}(\mathbf{x}_{\perp})] \right\}^N \\ &= \frac{2}{N!} \left(\frac{\beta}{2N} \right)^N \left\{ n\mathcal{N} + \frac{\mathcal{N}_{\perp}}{n^N} + \text{cross-term} \right\} \end{aligned} \quad (19)$$

which is subleading for $N \geq 3$. For SU(2) theory, we have an additional $O(\beta^2)$ contribution,

$$\ln Z_n^{(N=2)} = -\beta n\mathcal{N} - \frac{\beta}{n}\mathcal{N}_{\perp} + \frac{\beta^2}{2N^2}n\mathcal{N} + \frac{\beta^2}{2N^2}\frac{1}{n^2}\mathcal{N}_{\perp} + O(\beta^4). \quad (20)$$

For SU(3) theory, it is

$$\ln Z_n^{(N=3)} = -\beta n\mathcal{N} - \frac{\beta}{n}\mathcal{N}_{\perp} + \frac{\beta^2}{4N^2}n\mathcal{N} + \frac{\beta^2}{4N^2}\frac{1}{n^2}\mathcal{N}_{\perp} + \frac{\beta^3}{3!N^3}\frac{n}{2^2}\mathcal{N} + \frac{\beta^3}{3!N^3}\frac{1}{2^2n^3}\mathcal{N}_{\perp} + O(\beta^4). \quad (21)$$

Putting together (13) and (19), the Renyi entropies for SU(N) theory to up order $O(\beta^3)$ read

$$S_n^{(N=2)} = -\beta\mathcal{N}_{\perp}\frac{1+n}{n} + \frac{\beta^2\mathcal{N}_{\perp}}{2N^2} \left(\frac{1+n+n^2}{n^2} \right) + O(\beta^4), \quad (22)$$

$$S_n^{(N=3)} = -\beta\mathcal{N}_{\perp}\frac{1+n}{n} + \frac{\beta^2\mathcal{N}_{\perp}}{4N^2} \left(\frac{1+n+n^2}{n^2} \right) + \frac{\beta^3\mathcal{N}_{\perp}}{24N^3} \frac{(1+n)(1+n^2)}{n^3} + O(\beta^4), \quad (23)$$

$$S_n^{(N>3)} = -\beta\mathcal{N}_{\perp}\frac{1+n}{n} + \frac{\beta^2\mathcal{N}_{\perp}}{4N^2} \left(\frac{1+n+n^2}{n^2} \right) + O(\beta^4). \quad (24)$$

As a result, the entanglement entropy of SU(N) gauge theory in the strong coupling expansion is

$$S_{\text{EE}}^{(N=2)} = \frac{A_{\perp}}{a_{\perp}^2} \left[-\frac{4N^2}{\lambda} + 6\frac{N^2}{\lambda^2} + O\left(\frac{N^4}{\lambda^4}\right) \right] + \delta S_{\text{EE}}^{(N=2)}, \quad (25)$$

$$S_{\text{EE}}^{(N=3)} = \frac{A_{\perp}}{a_{\perp}^2} \left[-\frac{4N^2}{\lambda} + 3\frac{N^2}{\lambda^2} + \frac{4}{3}\frac{N^3}{\lambda^3} + O\left(\frac{N^4}{\lambda^4}\right) \right] + \delta S_{\text{EE}}^{(N=3)}, \quad (26)$$

$$S_{\text{EE}}^{(N>3)} = \frac{A_{\perp}}{a_{\perp}^2} \left[-\frac{4N^2}{\lambda} + 3\frac{N^2}{\lambda^2} + O\left(\frac{N^4}{\lambda^4}\right) \right] + \delta S_{\text{EE}}^{(N>3)}. \quad (27)$$

where A_{\perp} is the area of interface, a_{\perp} is lattice spacing in the transverse space and where $\lambda = g^2N$ is the t'Hooft coupling. As we argued below Eq. (12), this result is independent

of how the lattice is discretized, as long as the conical singularity is encircled by the same number of plaquettes in the action. The additional terms δS_{EE} are the positive contribution from the cosmological constant living on the $d - 1$ dimensional space transverse to the conical singularity in the Lagrangian, which we expect to cancel the negative term in the entanglement entropy, and will be explained in Sec. (IV).

Likewise, the $U(1)$ result can also be obtained using the same method:

$$S_{\text{EE}}^{(U(1))} = \frac{A_{\perp}}{a_{\perp}^2} \left[-\frac{4}{g^2} + \frac{3}{g^4} + O\left(\frac{1}{g^8}\right) \right] + \delta S_{\text{EE}}^{(U(1))}. \quad (28)$$

Note that this is for the Yang-Mills $U(1)$ gauge fields, instead of the electromagnetic $U(1)$.

IV. A 2-D COSMOLOGICAL CONSTANT COUNTERTERM

In the result Eqs. (25-27) and (28), one finds that the leading order $O(\beta)$ term in the entanglement entropy has a negative contribution while all the subleading orders are positive. It turns out that, with the 2 dimensional conical structure in a 4 dimensional space time, we are allowed to introduce more local operators in the continuum action

$$S = \int d^2x_{\perp} d^2x_{\parallel} \left[-\frac{1}{4} F^2 + c_4 + c_2 \delta^{(2)}(x_{\parallel}, \tau) \right]. \quad (29)$$

The c_4 term is a four dimensional cosmological constant counter term which does not contribute to the entanglement entropy. However, the two dimensional cosmological constant counter term c_2 , living on the space transverse to the conical singularity and breaks the translational symmetry on the cone, can contribute to the entanglement entropy [18]. Assuming c_2 is a smooth function of n , then

$$c_2 = c_2'(n - 1) + O((n - 1)^2), \quad (30)$$

where we $c_2' = c_2'(\beta)$ is a function of β . We have made use of the fact that c_2 should vanish at $n = 1$ where translational symmetry is recovered. Therefore there is an extra unspecified contribution to the entanglement entropy which also obeys the area law:

$$\delta S_{\text{EE}}^{(N)} = A_{\perp} c_2'^{(N)}. \quad (31)$$

We label the N dependence explicitly since different theories would have different c_2' counter terms. Also, $c_2'^{(N)}$ is β dependent.

The negative leading term in our entanglement entropy could in principle be compensated by the contribution from the two dimensional cosmological constant in (31). We remind the readers here that the negative leading term arises from the constant term in the lattice gauge field Lagrangian. Since the constant in the Lagrangian for the central plaquettes is different from that for the non-central ones by a factor of $1/n$, the effect is just like the c_2 in Eq.(29). If these constant terms are not included in the lattice Lagrangian, like what was done in some of the actions studied previously [15, 17, 20], then the negative contribution to the entanglement entropy will not arise, and the Lagrangian can not be reduced to the usual Yang-Mills one at continuum limit. We argue that the c_2 and c_4 terms will always appear by renormalization even they are set to zero at certain renormalization scale. Setting them to be zero is equivalent to choosing specific values for these counter terms. Previously it was known that different choices of the boundary conditions gave different values for the entanglement entropy [11]. This corresponds to employing different regularization schemes, but those differences can be compensated by having different values for the counter terms for different regularization schemes used.

We further speculate that the two-dimensional cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Also, they could play the role of a counter term to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

V. COMPARISON WITH PREVIOUS RESULTS

Our system is of infinite size in spatial and time dimensions. In [15], the entanglement entropy of $SU(N)$ gauge fields on the lattice of finite size at finite temperature is calculated in the Lagrangian formalism by means of group characteristic expansions. In 1+1 dimensions, when the periodic boundary condition is imposed on the spatial dimension, the entanglement entropy is given by

$$S_{\text{ent.}} = \left(\frac{\beta}{2N^2} \right)^{A/a^2} \left[1 - \log \left(\left(\frac{\beta}{2N^2} \right)^{A/a^2} / N^2 \right) \right] \quad (32)$$

where A is the total area of the 1+1 dimensional spacetime and a is the lattice spacing. At the limit of infinite area, the entanglement entropy reduces to 0, instead of Eqs. (25–27). While the free boundary condition is chosen in the spatial dimensions, the entanglement

entropy vanishes identically. We can obtain the same result if we set the conical singularity on a site so there is no central plaquette at all. So the leading order result for a system of periodic boundary condition in 1+1 dimensions comes from the configuration with the plaquettes tiling the whole dimensions. But this configuration gives vanishing entanglement entropy when the free spatial boundary condition is taken. Despite this, the result in Eq.(32) does not have the area law due to the nature of 1 spatial dimensional system, and hence does not have the expected form.

Ref. [10] considers the entanglement entropy of the ground states of the $SU(2)$ Kogut-Susskind Hamiltonian [21] for the Wilson gauge theories at strong coupling limit, by including the edge states living on the boundary into the Hilbert space, such that the total entanglement entropy contains the contribution from the edge states. For $d + 1 \geq 3$ dimensions, the leading order entanglement entropy obtained by [10] is

$$S_{\text{ent.}} = \frac{A_{\perp}}{a^2} (d-1) \beta^2 \left(\ln \frac{1}{\beta^2} + 1 + 2 \ln 2 \right) \quad (33)$$

where A_{\perp} is the boundary area [22]. The entire leading contribution to the entropy in (33) is given by the entanglement entropy of the edge modes, which corresponds to the contribution of the cosmological constant counterterm in our model. The non-local correlations of the d.o.f.'s in the two subregions are manifest only at higher order. Ref. [12] also gives similar result: it is demonstrated via numerical simulation that, in the case of the Z_2 lattice gauge theory in three dimensions, the entanglement entropy is almost saturated by the entropy of the end points of the electric strings cut open by the boundary of the two subregions. In our language, their setup corresponds to locating the conical singularity on a site, so there is no central plaquette at all such that the leading contribution is coming from the counterterm.

VI. SUMMARY

To summarize, we have calculated the entanglement entropy of the $SU(N)$ Yang-Mills gauge theories on the lattice under the strong coupling expansion in powers of $\beta = 2N/g^2$. Using the replica method, our Lagrangian formalism maintains gauge invariance on the lattice. At $O(\beta^2)$ and $O(\beta^3)$, the entanglement entropy is solely contributed by the central plaquettes enclosing the conical singularity of the n -sheeted Riemann surface. The area law emerges naturally to the highest order $O(\beta^3)$ of our calculation. The leading $O(\beta)$ term

is negative, which could in principle be canceled by taking into account the cosmological constant living in interface of the two entangled subregions. This unknown cosmological constant resembles the ambiguity of edge modes in the Hamiltonian formalism. We have further speculated that this unknown cosmological constant can show up in the entanglement entropy of scalar and spinor field theories as well. Furthermore, it could play the role of a counterterm to absorb the ultraviolet divergence of entanglement entropy and make entanglement entropy a finite physical quantity.

Acknowledgments

We would like to thank Michael Endres, David Lin, Feng-Li Lin, Masahiro Nozaki, Chen-Te Ma, Jackson Wu and Yun-Long Zhang for helpful discussions. This work is supported by the MOST, NTU-CTS and the NTU-CASTS of Taiwan. SHD is supported by Grant No. NSC103-2811-M-002-134. JYP is supported in part by NSFC under grant No. 11125524 and 1221504.

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- [22] Note that the β^2 in [10] is equivalent to our β . Eq. (33) is expressed in our notation.